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A unified proof on existence of homoclinic orbits for some semilinear ordinary differential equations with periodic potentials ☆

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ABSTRACT

This paper gives a direct, short and unified proof of Rabinowitz's Theorem, Grossinho–Minhós–Tersian's Theorem and Tersian–Chaparova's Theorems on existence of homoclinic orbits for second order periodic Hamiltonian systems, fourth and sixth order periodic ordinary differential equations, respectively, by Brezis–Nirenberg type Mountain Pass Lemma.

1. Introduction

Since 1990, there have been many literatures (cf. [1,4–14] and the references therein) on the subject of homoclinic orbits for differential equations by variational methods. In some situations, homoclinic orbits of these equations can be founded as critical points of the corresponding $\mathbb{C}^1$-functionals of the form

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} V\big(t, u(t)\big)\,\mathrm{d}t \tag{1.1}$$

in appropriate Hilbert spaces. In the present paper, we are mainly interested in the superquadratic potential $V(t,u)$, and assume that it satisfies:

(V_1) $V(t,u) \in \mathbb{C}^1(\mathbb{R}\times\mathbb{R})$ is T-periodic in the variable t,
(V_2) there exist constants $\alpha > 2$, $r > 0$ such that $\alpha V(t,u) \leqslant u V_u(t,u)$, $\forall t \in \mathbb{R}$, $u \in \mathbb{R}^n \backslash \{0\}$, and $V(t,u) > 0$, $\forall t \in \mathbb{R}$, $|u| \geqslant r > 0$,
(V_3) $V_u(t,u) = o(|u|)$ as $u \to 0$ uniformly in $t \in \mathbb{R}$,

where $V_u(t,u) = \partial V/\partial u$ is the gradient in the variable $u \in \mathbb{R}^n$. This is indeed the case for second order periodic Hamiltonian systems studied by Rabinowitz [11], Grossinho, Minhós and Tersian [7], for fourth and sixth order semilinear periodic ordinary differential equations considered by Tersian and Chaparova [13,14], and so on. A surprising fact is that four important theorems on existence of homoclinic orbits in [11,7,13,14] are included by our main result.

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Theorem 1.1. *Let $(E, \|\cdot\|)$ be a Hilbert space with the norm such that*

$$(*)\qquad \|u(.)\| = \|u(.+jT)\|, \quad \forall u \in E,\ j \in Z.$$

$\mathbb{C}_0^\infty(\mathbb{R}, \mathbb{R}^n)$ is dense in E, $u(\pm\infty) = 0\ \forall u \in E$, and E is embedded into $L^p(\mathbb{R}, \mathbb{R}^n)$ $(2 \leqslant p \leqslant \infty)$ continuously. Under the assumptions of (V_1)–(V_3), the functional $\varphi(u) \in \mathbb{C}^1(E, \mathbb{R})$ given in (1.1) has at least one nontrivial critical point in E.

Remark 1.2. (V_2) is the so-called Ambrosetti–Rabinowitz type superquadratic condition where the potential $V(t,u)$ can change sign, (V_3) implies that $u = 0$ is a trivial critical point of $\varphi(u)$, and (V_2) together with (V_3) shall show that $\varphi(u)$ possesses the mountain pass type geometry structure. However, since (V_1) and $(*)$, $\varphi(u)$ does not satisfy the Palais–Smale condition, that is:

(PS) Any sequence $\{u_m\} \subset E$ such that $\varphi(u_m)$ is bounded and $\varphi'(u_m) \to 0$ possesses a convergent subsequence.

Therefore, Theorem 1.1 cannot be proved by the following usual Mountain Pass Lemma due to Ambrosetti and Rabinowitz [2].

Theorem 1.3. *(See [2].) Let X be a Banach space and $\psi \in \mathbb{C}^1(X, \mathbb{R})$ with $\psi(0) = 0$. If ψ also satisfies* (PS) *and*:

(i) *There are constants $\mu > 0$ and $\rho > 0$ such that $\psi(u) \geqslant \mu$, $\forall \|u\| = \rho$.*
(ii) *There exists $e \in X \backslash B_\rho(0)$ such that $\psi(e) < 0$.*

Then ψ has a critical value $c^* \geqslant \mu$ where

$$c^* = \inf_{\gamma \in \Gamma} \max_{0 \leqslant s \leqslant 1} \psi\big(\gamma(s)\big) \geqslant \mu \tag{1.2}$$

with $\Gamma = \{\gamma \in \mathbb{C}([0,1], X)\colon\ \gamma(0) = 0,\ \gamma(1) = e\}$.

In Theorem 1.3, if the functional ψ does not satisfy (PS), then Brezis and Nirenberg has proved the fact that ψ has at least a $(PS)_{c^*}$ sequence, where c^* is defined in (1.2), namely, there exists a sequence $\{u_m\} \subset X$ such that $\psi(u_m) \to c^*$ and $\psi'(u_m) \to 0$. The result is so-called Brezis–Nirenberg type Mountain Pass Lemma [3].

Remark 1.4. There exist Hilbert spaces E embedded into $L^p(\mathbb{R}, \mathbb{R}^n)$ $(2 \leqslant p \leqslant \infty)$ whose norms $\|.\|_E$ do not satisfy the assumption $(*)$. For instance, the Hilbert space $E_0 = \{u \in H^1(\mathbb{R})\colon\ \int_{\mathbb{R}} (|\dot{u}(t)|^2 + e^{c|t|}|u(t)|^2)\,dt < \infty\}$, $c > 0$, with the norm $\|u\|_0 = (\int_{\mathbb{R}} (|\dot{u}(t)|^2 + e^{c|t|}|u(t)|^2)\,dt)^{1/2}$, embedded into $L^p(\mathbb{R})$ $(2 \leqslant p \leqslant \infty)$ [10]. Let us take a continuous function $u_0 \in E_0$ such that $u_0(t) > 0$ if $t \in (0,1)$ and $u_0(t) = 0$ if t is not in $(0,1)$. Then, if $j < 0$, $T > 0$, $\|u_0(.+jT)\|_0 \geqslant (e^{-jTc/2})(\int_{\mathbb{R}} (e^{c|t|}|u_0(t)|^2)\,dt)^{1/2} \to \infty$ as $j \to -\infty$.

The organization of this paper is as follows. In Section 2, the proof of Theorem 1.1 is presented. In Section 3, firstly, we use our Theorem 1.1 to prove Rabinowitz's Theorem on existence of homoclinic orbits for second order periodic Hamiltonian systems [11]

$$\ddot{u}(t) - A(t)u(t) + V_u\big(t, u(t)\big) = 0 \tag{HS}$$

together with Grossinho–Minhós–Tersian's Theorem [7] on positive homoclinic orbits for second order periodic scalar differential equations; secondly, in view of Theorem 1.1, we prove Tersian–Chaparova's Theorems (see [13,14]) on existence of homoclinic orbits for fourth and sixth order periodic scalar differential equations

$$u^{(4)}(t) + p\ddot{u}(t) + a(t)u(t) - b(t)u(t)^2 - c(t)u(t)^3 = 0, \quad -\infty < t < \infty, \tag{FE}$$

$$u^{(6)}(t) + Du^{(4)}(t) + B\ddot{u}(t) - u(t) + a(t)u(t)|u(t)|^\sigma = 0, \quad -\infty < t < \infty. \tag{SE}$$

Moreover, we also obtain some more general conclusions, especially, we prove that one important condition in Grossinho–Minhós–Tersian's Theorem is not necessary.

2. Proof of Theorem 1.1

Some of the ideas of this section were used before (e.g. [5,10]), and we give them only for completeness.

Lemma 2.1. *If (V_1)–(V_3) hold, then there exist $\bar{\mu} > 0$ and $\bar{\rho} > 0$ such that $\varphi(u) \geqslant \bar{\mu}$, $\forall \|u\| = \bar{\rho}$, and there is $\bar{e} \in X \backslash B_{\bar{\rho}(0)}$ such that $\varphi(\bar{e}) < 0$.*

Proof. Embedding inequalities imply that there are constants $k_1, k_2 > 0$ such that $\|u\|_{L^2(\mathbb{R})} \leqslant k_1\|u\|$, $\|u\|_{L^\infty(\mathbb{R})} \leqslant k_2\|u\|$, $\forall u \in E$. In view of (V_3), there exists $\delta \in (0,1)$ such that $|V(t,u)| \leqslant |u|^2/4k_1^2$, $\forall |u| \leqslant \delta$, uniformly in $t \in \mathbb{R}$. Choosing $\bar\rho = \delta/k_2$, and $\|u\| = \bar\rho$, we conclude that $|u(t)| \leqslant \delta$, $\forall t \in \mathbb{R}$. Thus we obtain that $\int_{\mathbb{R}} V(t,u(t))\,dt \leqslant \|u\|^2_{L^2(\mathbb{R})}/4k_1^2 \leqslant \|u\|^2/4$, and $\varphi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} V(t,u(t))\,dt \geqslant \|u\|^2/4 = \bar\rho^2/4 \equiv \bar\mu$. On the other hand, by (V_2), there exist $k_3, k_4 > 0$ such that $V(t,u) \geqslant k_3|u|^\alpha - k_4$, $\forall t \in \mathbb{R}$, $u \in \mathbb{R}^n$. Taking $e_1 = (1,0,0,\dots,0) \in \mathbb{R}^n$ and $g(t) \in \mathbb{C}_0^\infty(\mathbb{R},\mathbb{R})$ such that $g(t) = 0$ if $t \leqslant 0$ or $t \geqslant 1$, and $g(t) > 0$ if $0 < t < 1$, we have for $\lambda \geqslant 1$,

$$\varphi(\lambda g e_1) = \frac{1}{2}\|\lambda g e_1\|^2 - \int_{\mathbb{R}} V\big(t, \lambda g(t) e_1\big)\,dt = \frac{1}{2}\|\lambda g e_1\|^2 - \int_{[0,1]} V\big(t, \lambda g(t) e_1\big)\,dt$$
$$\leqslant \frac{1}{2}\|\lambda g e_1\|^2 - \int_{[0,1]} \big(k_3\big|\lambda g(t) e_1\big|^\alpha - k_4\big)\,dt = \frac{\lambda^2}{2}\|g e_1\|^2 - k_3\lambda^\alpha \int_{[0,1]} \big|g(t)\big|^\alpha\,dt + k_4 \to -\infty \tag{2.1}$$

as $\lambda \to \infty$. So we can choose $\bar e = \lambda g e_1$ for λ large. □

Lemma 2.2. *If (V_1)–(V_3) hold, then there exists $\bar c > 0$ such that φ has a bounded $(PS)_{\bar c}$ sequence.*

Proof. By Lemma 2.1 and Remark 1.2, then there exists $\bar c > 0$ such that φ has a $(PS)_{\bar c}$ sequence $\{u_m\}$ satisfying $\varphi(u_m) \to \bar c$ and $\varphi'(u_m) \to 0$, where

$$\bar c = \inf_{\gamma \in \Gamma} \max_{0 \leqslant s \leqslant 1} \varphi\big(\gamma(s)\big) \geqslant \bar\mu > 0 \tag{2.2}$$

with $\Gamma = \{\gamma \in \mathbb{C}([0,1],E)\colon\ \gamma(0) = 0,\ \gamma(1) = \bar e\}$. Using (V_2), we infer that

$$\alpha\varphi(u_m) - \varphi'(u_m)u_m = \left(\frac{\alpha}{2} - 1\right)\|u_m\|^2 - \int_{\mathbb{R}} \big(\alpha V\big(t, u_m(t)\big) - V_u\big(t, u_m(t)\big)u_m(t)\big)\,dt \geqslant \left(\frac{\alpha}{2} - 1\right)\|u_m\|^2. \tag{2.3}$$

Since $\alpha > 2$, (2.3) implies the boundedness of $\{u_m\}$. □

Proof of Theorem 1.1. For any $m \in N$, there is $j_m \in \mathbb{Z}$ such that the maximum of $|u_m(t + j_m T)|$ occurs in $[0,T]$. Let $w_m(t) = u_m(t + j_m t)$, then $\|w_m\| = \|u_m\|$ and $\varphi(w_m) = \varphi(u_m)$. Without loss of generality, we may assume that $w_m \to w$ weakly in E, $w_m \to w$ in $L^2_{\mathrm{loc}}(\mathbb{R})$, and $w_m \to w$ in $\mathbb{C}_{\mathrm{loc}}(\mathbb{R})$. Thus for any $\upsilon \in \mathbb{C}_0^\infty(\mathbb{R})$, since (V_1) and $(*)$, we have

$$\big|\varphi'(w_m)\upsilon(\cdot)\big| = \big|\varphi'(u_m)\upsilon(\cdot - j_m T)\big| \leqslant \big\|\varphi'(u_m)\big\|\,\big\|\upsilon(\cdot - j_m T)\big\| = \big\|\varphi'(u_m)\big\|\,\big\|\upsilon(\cdot)\big\| \to 0. \tag{2.4}$$

That is $(w_m, \upsilon) - \int_{\mathbb{R}} V_u(t, w_m(t))\upsilon(t)\,dt \to 0$. Considering $\upsilon \in \mathbb{C}_0^\infty(\mathbb{R},\mathbb{R}^n)$, we get

$$\varphi'(w)\upsilon = (w, \upsilon) - \int_{\mathbb{R}} V_u\big(t, w(t)\big)\upsilon(t)\,dt = 0. \tag{2.5}$$

Since $\mathbb{C}_0^\infty(\mathbb{R},\mathbb{R}^n)$ is dense in E, $w \in E$ is a critical point of φ. Lastly, we must show that $w \neq 0$. If this is not true, we have

$$\|u_m\|_{L^\infty(\mathbb{R})} = \|w_m\|_{L^\infty(\mathbb{R})} = \|w_m\|_{L^\infty([0,T])} \to 0. \tag{2.6}$$

Thus by (2.6) and (V_3), given $\varepsilon > 0$, we have for any $t \in \mathbb{R}$ and m sufficiently large

$$\big|V\big(t, u_m(t)\big)\big| \leqslant \varepsilon\big|u_m(t)\big|^2, \qquad \big|u_m(t) V_u\big(t, u_m(t)\big)\big| \leqslant \varepsilon\big|u_m(t)\big|^2. \tag{2.7}$$

So, in view of (2.7), for m large, we conclude that

$$0 < 2\varphi(u_m) = \varphi'(u_m)u_m - \int_{\mathbb{R}} \big(2V\big(t, u_m(t)\big) - V_u\big(t, u_m(t)\big)u_m(t)\big)\,dt$$
$$\leqslant \varphi'(u_m)u_m + 3\varepsilon \int_{\mathbb{R}} \big|u_m(t)\big|^2\,dt \leqslant \big\|\varphi'(u_m)\big\|\,\|u_m\| + 3\varepsilon k_1^2\|u_m\|^2. \tag{2.8}$$

Since $\varphi'(u_m) \to 0$, $\|u_m\|$ is bounded and ε is arbitrary, (2.8) implies that $\varphi(u_m) \to 0$, which is contradiction with the fact $\varphi(u_m) \to \bar c \geqslant \bar\mu > 0$ by Lemma 2.2. So $w \neq 0$. The proof is complete. □

3. A unified proof on four classical theorems

(A) Second order periodic Hamiltonian systems

In 1990, in [11], Rabinowitz discussed the existence of homoclinic orbits for the second order Hamiltonian system

$$\ddot{u}(t) - A(t)u(t) + V_u\big(t, u(t)\big) = 0 \tag{HS}$$

where $u = u(t) = (u_1(t), u_2(t), \dots, u_n(t)) \in \mathbb{R}^n$, $A(t) \in \mathbb{C}(\mathbb{R}, \mathbb{R}^{n^2})$ is a real symmetric $n \times n$ matrix, $V(t,u) \in \mathbb{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. He gave the following result:

Theorem 3.1 *(Rabinowitz's Theorem). (See [11].) Assume that $A(t)$ is positive and T-periodic, $V(t,u)$ satisfies (V_1), (V_3) and*

(V_2') *There exists a constant $\alpha > 2$ such that* $0 < \alpha V(t,u) \leqslant u V_u(t,u)$, $\forall t \in \mathbb{R}$, $u \in \mathbb{R}^n \backslash \{0\}$.

Then Eq. (HS) *possesses at least one nontrivial homoclinic orbit.*

In order to prove Theorem 3.1, Rabinowitz considered the approximating periodic problem

$$\begin{cases} \ddot{u} - A(t)u(t) + V_u(t, u(t)) = 0, & t \in (-kT, kT), \\ u(-kT) = u(kT), & \dot{u}(-kT) = \dot{u}(kT), \end{cases} \tag{3.1}$$

for any $k \in \mathbb{N}$, and let

$$E_k = H^1_{2kT}\big(\mathbb{R}, \mathbb{R}^n\big) = \left\{ u\colon u \text{ is } 2kT\text{-periodic in } \mathbb{R} \text{ such that } \int_{-kT}^{kT} \left(\left|\dot{u}(t)\right|^2 + \left|u(t)\right|^2\right) \mathrm{d}t < \infty \right\}, \tag{3.2}$$

$$\Phi_k(u) = \frac{1}{2} \int_{-kT}^{kT} \left(\left|\dot{u}(t)\right|^2 + \big(A(t)u(t), u(t)\big)\right) \mathrm{d}t - \int_{-kT}^{kT} V\big(t, u(t)\big)\, \mathrm{d}t, \quad u \in E_k. \tag{3.3}$$

It is verified easily that Φ_k satisfies (PS). By Theorem 1.3, Φ_k has a critical point $e_k \in E_k$. Then he obtained one homoclinic orbit of (HS) as the limit of $2kT$-periodic extensions of $\{e_k\}$ by variational characterization of the corresponding critical values.

Now, we shall use our Theorem 1.1 to prove Theorem 3.1 directly. Denoted by $E = H^1(\mathbb{R})$ the usual Sobolev space with the norm $\|u\|_E = (\int_{\mathbb{R}} (|\dot{u}(t)|^2 + |u(t)|^2)\, \mathrm{d}t)^{\frac{1}{2}}$. Since $A(t)$ is positive and periodic,

$$\|u\| = \left(\int_{\mathbb{R}} \left(\left|\dot{u}(t)\right|^2 + \big(A(t)u(t), u(t)\big)\right) \mathrm{d}t \right)^{\frac{1}{2}}$$

is also a norm of E, equivalent to $\|u\|_E$. Define

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}} \left(\left|\dot{u}(t)\right|^2 + \big(A(t)u(t), u(t)\big)\right) \mathrm{d}t - \int_{\mathbb{R}} V\big(t, u(t)\big)\, \mathrm{d}t = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} V\big(t, u(t)\big)\, \mathrm{d}t, \quad u \in E. \tag{3.4}$$

Using (V_1), (V_2') and (V_3), by [6], we know that $\Phi \in \mathbb{C}^1(E)$, and the critical points of Φ in E are homoclinic orbits of (HS). And since (V_2') implies (V_2), the assumption $(*)$ is also satisfied, by Theorem 1.1, we derive that the functional Φ has a nontrivial critical point in E, which is a nontrivial homoclinic orbit of (HS).

We should point out that, Rabinowitz's approximating method is a useful tool, and has been widely used in the field of homoclinic orbits of Eq. (HS). Particularly, in 1999, using the above trick, Grossinho, Minhós and Tersian [7] considered a scalar case of Eq. (HS), that is

$$\ddot{u}(t) - a(t)u(t) + b(t)u(t)^2 + c(t)u(t)^3 = 0, \quad -\infty < t < \infty, \tag{SHS}$$

where the coefficients $a(t)$, $b(t)$ and $c(t)$ are T-periodic continuous functions such that

$$0 < a_1 \leqslant a(t) \leqslant a_2, \qquad 0 \leqslant b_1 \leqslant b(t) \leqslant b_2, \qquad 0 < c_1 \leqslant c(t) \leqslant c_2, \tag{3.5}$$

$$b_2^2 - b_1^2 < 4a_1c_2, \tag{3.6}$$

and they gave a more attractive result than Theorem 3.1.

Theorem 3.2 *(Grossinho–Minhós–Tersian's Theorem). (See [7].) Let the assumptions* (3.5) *and* (3.6) *hold. Then Eq.* (SHS) *has a positive homoclinic orbit.*

In order to prove Theorem 3.2 directly, we need a variant of Theorem 1.1, we have:

Theorem 3.3. *The conclusion in Theorem* 1.1 *still holds if $n = 1$, and we replace Hypothesis (V_2) with*:

(V_2') *There exist constants $\alpha > 2$ and $r > 0$ such that $\alpha V(t,u) \leqslant uV_u(t,u)$, $\forall t \in \mathbb{R}$, $u \in \mathbb{R}\backslash\{0\}$ and $V(t,u) > 0$, $\forall t \in \mathbb{R}$, $u \geqslant r > 0$.*

Indeed, proof of Theorem 3.3 is the same as that of Theorem 1.1, so we omit it. Now we consider the modified equation

$$\ddot{u}(t) - a(t)u(t) + b(t)u(t)^2 + c(t)u(t)_+^3 = 0, \quad -\infty < t < \infty, \tag{SHS+}$$

where $u_+ = \max(u, 0)$, and define the associated functional

$$\begin{aligned}\Psi(u) &= \frac{1}{2}\int_{\mathbb{R}} \left(\left|\dot{u}(t)\right|^2 + a(t)\left|u(t)\right|^2\right) \mathrm{d}t - \int_{\mathbb{R}} \left(\frac{1}{3}b(t)\big(u(t)\big)^3 + \frac{1}{4}c(t)\big(u(t)_+\big)^4\right) \mathrm{d}t \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \left(\frac{1}{3}b(t)\big(u(t)\big)^3 + \frac{1}{4}c(t)\big(u(t)_+\big)^4\right) \mathrm{d}t, \quad u \in E = H^1(\mathbb{R}),\end{aligned} \tag{3.7}$$

where $V(t,u) = \frac{1}{3}b(t)u^3 + \frac{1}{4}c(t)u_+^4$ such that $3V(t,u) \leqslant uV_u(t,u)$, $\forall t \in \mathbb{R}$, $u \in \mathbb{R}\backslash\{0\}$, and $V(t,u) \to \infty$ as $u \to \infty$ uniformly in $t \in \mathbb{R}$. Thus, by Theorem 3.3, $\Psi(u)$ has one nontrivial critical point $u = u(t) \in H^1(\mathbb{R})$, which is one homoclinic orbit of Eq. (SHS+). Further, multiplying equation (SHS+) by $u_-(t)$, where $u_- = \min(u, 0)$, and integrating over $\mathbb{R}$, we find

$$0 \leqslant \int_{\mathbb{R}} \left(\left|\dot{u}_-(t)\right|^2 + a(t)\left|u_-(t)\right|^2\right) \mathrm{d}t = \int_{\mathbb{R}} b(t)u(t)^2 u_-(t)\, \mathrm{d}t \leqslant 0 \tag{3.8}$$

since $0 \leqslant b_1 \leqslant b(t) \leqslant b_2$. Clearly, (3.8) implies that $\|u_-(t)\| = 0$, $u_-(t) = 0$. So, we get

$$u(t) = u_+(t) + u_-(t) = u_+(t) \geqslant 0.$$

Finally, by the initial theorem of ordinary differential equations, we have $u(t) > 0$, $\forall t \in \mathbb{R}$. Thus, $u(t)$ is a positive homoclinic orbit of Eq. (SHS).

Remark 3.4. By our proof above, it can be seen that the inequality condition (3.6) in Theorem 3.2 is not necessary.

(B) Fourth order semilinear ordinary differential equations

In 2001, in [13], Tersian and Chaparova studied the existence of homoclinic orbits of a class of fourth order scalar semilinear differential equations

$$u^{(4)}(t) + p\ddot{u}(t) + a(t)u(t) - b(t)u(t)^2 - c(t)u(t)^3 = 0, \quad -\infty < t < \infty, \tag{FE}$$

where $a(t)$, $b(t)$ and $c(t)$ are T-periodic continuous functions such that

$$0 < a_1 \leqslant a(t) \leqslant a_2, \qquad p < 2\sqrt{a_1}, \qquad \left|b(t)\right| \leqslant b, \qquad 0 < c_1 \leqslant c(t) \leqslant c_2. \tag{3.9}$$

This kind of problem together with Case (C) below is usually related to the existence of solitary waves or to the existence of stationary solutions with finite energy. Denoted by $E = H^2(\mathbb{R})$ the Sobolev space with the norm $\|u\|_E = (\int_{\mathbb{R}}(|\ddot{u}(t)|^2 + |\dot{u}(t)|^2 + |u(t)|^2)\,\mathrm{d}t)^{\frac{1}{2}}$, and, under the assumption of $p < 2\sqrt{a_1}$, they proved that the following inequality in E:

$$\int_{\mathbb{R}} \left(\left|\ddot{u}(t)\right|^2 - p\left|\dot{u}(t)\right|^2 + a(t)u(t)^2\right) \mathrm{d}t \geqslant M \int_{\mathbb{R}} \left(\left|\ddot{u}(t)\right|^2 + \left|\dot{u}(t)\right|^2 + \left|u(t)\right|^2\right) \mathrm{d}t \tag{3.10}$$

holds for some $M > 0$. Thus the norm $\|u\| = (\int_{\mathbb{R}}(|\ddot{u}(t)|^2 - p|\dot{u}(t)|^2 + a(t)u(t)^2)\,\mathrm{d}t)^{\frac{1}{2}}$ is equivalent to $\|u\|_E$. Clearly the homoclinic orbits of (FE) can be found as critical points of the functional

$$\begin{aligned}I(u) &= \frac{1}{2}\int_{\mathbb{R}} \left(\left|\ddot{u}(t)\right|^2 - p\left|\dot{u}(t)\right|^2 + a(t)u(t)^2\right) \mathrm{d}t - \int_{\mathbb{R}} \left(\frac{1}{3}b(t)\big(u(t)\big)^3 + \frac{1}{4}c(t)\big(u(t)\big)^4\right) \mathrm{d}t \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \left(\frac{1}{3}b(t)\big(u(t)\big)^3 + \frac{1}{4}c(t)\big(u(t)\big)^4\right) \mathrm{d}t, \quad u \in E.\end{aligned} \tag{3.11}$$

After proving that the functional $I(u)$ satisfies the two geometric conditions of Mountain Pass Lemma, and finding a Palais–Smale sequence $\{u_m\}$ in $H^2(\mathbb{R})$, they applied the Concentration–Compactness [4] argument, a rather hard trick, and obtained the conclusion as follows:

Theorem 3.5 *(Tersian–Chaparova's Theorem). (See [13].) Suppose that $a(t)$, $b(t)$ and $c(t)$ are T-periodic continuous functions. Under the condition* (3.9)*, there exists a nontrivial homoclinic orbit $u \in H^2(\mathbb{R})$ of Eq.* (FE).

Now, let us solve this problem again. Under the case of (FE), $V(t,u) = \frac{1}{3}b(t)u^3 + \frac{1}{4}c(t)u^4$ such that $3V(t,u) \leqslant uV_u(t,u)$, $\forall t \in \mathbb{R}$, $u \in \mathbb{R}\backslash\{0\}$, and $V(t,u) \to \infty$ as $|u| \to \infty$ uniformly in $t \in \mathbb{R}$. Therefore, in view of (3.11) and our Theorem 1.1, Theorem 3.5 is clearly true. Moreover, we have the following more general result:

Corollary 3.6. *Suppose that $a(t)$ is T-periodic continuous function such that $0 < a_1 \leqslant a(t) \leqslant a_2$, $p < 2\sqrt{a_1}$. Assume that $V(t,u)$ satisfies (V_1)–(V_3), then there exists a nontrivial homoclinic orbit $u \in H^2(\mathbb{R})$ of the equation*

$$u^{(4)}(t) + p\ddot{u}(t) + a(t)u(t) - V_u\big(t, u(t)\big) = 0, \quad -\infty < t < \infty. \tag{3.12}$$

In fact, Eq. (3.12) generalizes Fisher–Kolmogorov equation [13].

(C) Sixth order semilinear ordinary differential equations

In 2002, in [14], Tersian and Chaparova also discussed the existence of homoclinic solutions of a class of sixth order scalar semilinear differential equations

$$u^{(6)}(t) + Du^{(4)}(t) + B\ddot{u}(t) - u(t) + a(t)u(t)\big|u(t)\big|^{\sigma} = 0, \quad -\infty < t < \infty, \tag{SE}$$

where $\sigma > 0$, the coefficients D and B are such that $D > 0$, $D^2 < 4B$, and $a(t)$ is a T-periodic continuous function such that $0 < a_1 \leqslant a(t) \leqslant a_2$. Denoted by $E = H^3(\mathbb{R})$ the Sobolev space with the norm $\|u\|_E = (\int_{\mathbb{R}}(|u^{(3)}(t)|^2 + |\ddot{u}(t)|^2 + |\dot{u}(t)|^2 + |u(t)|^2)\,\mathrm{d}t)^{\frac{1}{2}}$. Using the assumptions $D > 0$, $D^2 < 4B$, Tersian and Chaparova proved the inequality

$$\int_{\mathbb{R}} \big(\big|u^{(3)}(t)\big|^2 - D\big|\ddot{u}(t)\big|^3 + B\big|\dot{u}(t)\big|^2 + \big|u(t)\big|^2\big)\,\mathrm{d}t \geqslant M' \int_{\mathbb{R}} \big(\big|u^{(3)}(t)\big|^2 + \big|\ddot{u}(t)\big|^3 + \big|\dot{u}(t)\big|^2 + \big|u(t)\big|^2\big)\,\mathrm{d}t$$

holds in E for some $M' > 0$, and homoclinic orbits of (SE) could be found as critical points of the functional

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}} \big(\big|u^{(3)}(t)\big|^2 - D\big|\ddot{u}(t)\big|^3 + B\big|\dot{u}(t)\big|^2 + \big|u(t)\big|^2\big)\,\mathrm{d}t - \frac{1}{\sigma+2}\int_{\mathbb{R}} a(t)\big|u(t)\big|^{\sigma+2}\,\mathrm{d}t \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{\sigma+2}\int_{\mathbb{R}} a(t)\big|u(t)\big|^{\sigma+2}\,\mathrm{d}t, \quad u \in E, \end{aligned} \tag{3.13}$$

where $\|u\| = (\int_{\mathbb{R}}(|u^{(3)}(t)|^2 - D|\ddot{u}(t)|^3 + B|\dot{u}(t)|^2 + |u(t)|^2)\,\mathrm{d}t)^{\frac{1}{2}}$ is equivalent to $\|u\|_E$. Still using Brezis–Nirenberg type Mountain Pass Lemma and Concentration–Compactness method, they proved the following theorem:

Theorem 3.7 *(Tersian–Chaparova's Theorem). (See [14].) Suppose that $\sigma > 0$, the coefficients D and B are such that $D > 0$, $D^2 < 4B$, and $a(t)$ is a T-periodic continuous function such that $0 < a_1 \leqslant a(t) \leqslant a_2$, then there exists a nontrivial homoclinic orbit $u \in H^3(\mathbb{R})$ of Eq.* (SE).

Actually, if we take $V(t,u) = a(t)|u|^{\sigma+2}/(\sigma+2)$, then for some $\beta \in (2, \sigma+2)$, we have $0 < \beta V(t,u) \leqslant uV_u(t,u)$, $\forall t \in \mathbb{R}$, $u \in \mathbb{R}\backslash\{0\}$, and $V(t,u) \geqslant a_1|u|^{\sigma+2}/(\sigma+2) \to \infty$ as $|u| \to \infty$, uniformly in $t \in \mathbb{R}$. Therefore, Theorem 3.7 is implied immediately by Theorem 1.1.

Corollary 3.8. *Suppose that $\sigma > 0$, the coefficients D and B are such that $D > 0$, $D^2 < 4B$. Assume that $V(t,u)$ satisfies (V_1)–(V_3), then there exists a nontrivial homoclinic orbit $u \in H^3(\mathbb{R})$ of the equation*

$$u^{(6)}(t) + Du^{(4)}(t) + B\ddot{u}(t) - u(t) + V_u\big(t, u(t)\big) = 0, \quad -\infty < t < \infty. \tag{3.14}$$

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